PERGAMON

Applied Mathematics Letters 13 (2000) 51–57

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Global Regularity of 3D Rotating Navier-Stokes Equations for Resonant Domains

A. BABIN
Department of Mathematics
University of California, Irvine, CA, 92697, U.S.A.

A. MAHALOV AND B. NICOLAENKO
Department of Mathematics
Arizona State University, Tempe, AZ 85287, U.S.A.

(Received May 1999; accepted June 1999)

Communicated by C. Bardos

Abstract—We prove existence on infinite time intervals of regular solutions to the 3D rotating Navier-Stokes equations in the limit of strong rotation (large Coriolis parameter Ω). This uniform existence is proven for periodic or stress-free boundary conditions for all domain aspect ratios, including the case of three wave resonances which yield nonlinear "2 1/2-dimensional" limit equations; smoothness assumptions are the same as for local existence theorems. The global existence is proven using techniques of the Littlewood-Paley dyadic decomposition. Infinite time regularity for solutions of the 3D rotating Navier-Stokes equations is obtained by bootstrapping from global regularity of the limit equations and convergence theorems. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Three-dimensional Navier-Strokes equations, Resonances, Rotation.

1. INTRODUCTION

We consider the three-dimensional Navier-Stokes equations in a frame rotating with an angular velocity $\Omega/2$ around the vertical axis e_3

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\,\mathbf{U} - \nu\Delta\mathbf{U} + \Omega \mathbf{e}_3 \times \mathbf{U} = -\nabla p + \mathbf{F}, \qquad \nabla\cdot\mathbf{U} = 0, \tag{1}$$

where $\mathbf{U}(t,x) = (U_1,U_2,U_3)$, $x=(x_1,x_2,x_3)$ is the velocity field, $\mathbf{F}$ is a divergence free force and p is the pressure. We consider regular solutions of equation (1), periodic with periods $2\pi a_j$ along the coordinate axes e_j $(j=1,2,3)$. We also consider stress-free boundary conditions with $U_3 = 0$, $\frac{\partial U_1}{\partial x_3} = \frac{\partial U_2}{\partial x_3} = 0$ for $x_3 = 0, 2\pi a_3$. In equation (1), e_3 denotes the vertical unit vector. We use Fourier series to represent physical fields:

$$\mathbf{U} = \sum_n \exp\left(i\left(\frac{n_1x_1}{a_1} + \frac{n_2x_2}{a_2} + \frac{n_3x_3}{a_3}\right)\right), \qquad \mathbf{U}_n = \sum_n \exp\left(i\tilde{n}\cdot x\right)\mathbf{U}_n, \tag{2}$$

PII: S0893-9659(99)00208-6

where $\mathbf{U}_n$ are the Fourier coefficients, $n = [n_1, n_2, n_3] \in \mathbf{Z}^3$, $\check{n} = (n_1/a_1, n_2/a_2, n_3/a_3)$. For stress-free boundary conditions we only need to restrict Fourier series to be even in x_3 for U_1, U_2 and odd in x_3 for U_3. As usual, all physical fields are taken to have zero average over the domain. In particular, $\mathbf{U}_{(0,0,0)} = 0$ in (2). We introduce the Sobolev spaces of periodic functions H^s with zero average and with the norm defined by $\|\mathbf{U}\|^2_{H^s} = \sum_n |\check{n}|^{2s}|\mathbf{U}_n|^2$, where $|\check{n}|^2 = \theta_1 n_1^2 + \theta_2 n_2^2 + \theta_3 n_3^2$, $\theta_j = 1/a_j^2$ $(j = 1, 2, 3)$. In this letter, we use the decomposition $\mathbf{U} = \bar{\mathbf{U}} + \mathbf{U}_{\mathrm{dp}}$, where $\bar{\mathbf{U}}(t, x_1, x_2)$ is the barotropic projection (vertical averaging), $\bar{\mathbf{U}}(t, x_1, x_2) = (1/2\pi a_3)\int_0^{2\pi a_3} \mathbf{U}(t, x_1, x_2, x_3)\, dx_3$ and the orthogonal field $\mathbf{U}_{\mathrm{dp}}(t, x_1, x_2, x_3)$ verifies $\bar{\mathbf{U}}_{\mathrm{dp}} = 0$. We denote $\mathbf{B}(\mathbf{U}, \mathbf{U}) = -\mathbf{P}(\mathbf{U} \cdot \nabla\mathbf{U})$, where $\mathbf{P}$ is the Leray projection on divergence free vector fields.

The linearized version of (1) was studied by Sobolev [1] who continued the analysis of Poincaré [2] (cf. [3]). The extension of this analysis to the genuine nonlinear equations (1) was done by Babin, Mahalov, and Nicolaenko [4–7] (henceforth BMN) utilizing sharp tools of small denominators and the Diophantine incommensurability conditions on the domain geometrical parameters a_1, a_2, a_3.

In this letter, we prove uniform existence in infinite time of regular strong solutions of equation (1) in the limit of fast but finite rotation, $\Omega \gg 1$. This result holds for all domain parameters a_1, a_2, a_3 including the case of three wave resonances for Poincaré inertial waves; such resonances yield strongly nonlinear "2 1/2-dimensional" limit equations. These limit equations are in fact for three-dimensional fields depending on x_1, x_2, and x_3, but with restricted wave-number interactions in the nonlinear term. The global existence is proven using techniques of the Littlewood-Paley dyadic decomposition. The limit $\Omega \to +\infty$ is a singular limit of fast oscillating solutions. BMN [7] demonstrate that such a limit is totally discontinuous as a function of the parameters a_1, a_2, a_3; the asymptotic nonlinear limit operator of dimension "2 1/2" only appears for a set of null Lebesgue measure for (a_1, a_2, a_3) in $\mathbf{R}^3$. In this letter, we establish infinite time regularity theorems valid for all domain parameters. Infinite time regularity for solutions of the 3D rotating Navier-Stokes equations (1) for Ω large but finite is obtained by bootstrapping from global regularity of the limit "2 1/2-dimensional" equations and convergence theorems including the case of domains resonant in a_1, a_2, a_3.

The main result is the following.

THEOREM 1. *Let a_1, a_2, a_3 be fixed but arbitrary. Let $\nu > 0$, $\alpha > 1/2$ and conditions of Theorem 4 on the force be satisfied. Let $\|\mathbf{U}(0)\|_0 \le M_0$, $\hat{T} = \|\mathbf{U}(0)\|_0^2/(\nu\lambda_1)$, where λ_1 is the first eigenvalue of the Stokes operator. Then for every $\Omega \ge \Omega'(a_1, a_2, a_3, \nu, M_{F_\alpha})$, Ω' independent of M_0, and for every weak solution $\mathbf{U}(t, x_1, x_2, x_3)$ of the three-dimensional rotating Navier-Stokes equations (1) defined on $[0, \hat{T}]$ which satisfies the classical energy estimates on $[0, \hat{T}]$, the following holds: $\mathbf{U}(t, x_1, x_2, x_3)$ can be extended to $0 < t < +\infty$ and it is regular for every $t : \hat{T} \le t < +\infty$; $\mathbf{U}(t, x_1, x_2, x_3)$ belongs to H^α and $\|\mathbf{U}(t, x_1, x_2, x_3)\|_\alpha \le C_1(a_1, a_2, a_3, M_{F_\alpha}, \nu)$ for every $t \ge \hat{T}$, where M_{F_α} is the H^α-norm of $\mathbf{F}$. If $\mathbf{F}$ is independent of t, then there exists a global attractor for the three-dimensional rotating Navier-Stokes equations bounded in H^α; such an attractor has a finite fractal dimension and attracts every weak Leray solution as $t \to +\infty$.*

2. GLOBAL EXISTENCE OF STRONG SOLUTIONS FOR 3D ROTATING NAVIER-STOKES EQUATIONS

We briefly recall the principle of averaging equation (1) over the fast time scale of Poincaré inertial waves [4–7]. The linear propagator $\mathbf{E}(-\Omega t) : \mathbf{U}(0) \to \mathbf{U}(t)$ solves the linear problem

$$\partial_t \mathbf{U} + \Omega\mathbf{PJPU} = 0, \qquad \mathbf{JU} = e_3 \times \mathbf{U}, \tag{3}$$

where $\mathbf{P}$ is the Leray projection. The eigenvalues of $\mathbf{PJP}$ are $\pm i\xi_n$, where

$$\xi_n = \frac{\check{n}_3}{|\check{n}|}, \quad \check{n}_3 = \frac{n_3}{a_3}, \quad |\check{n}| = \sqrt{\theta_1 n_1^2 + \theta_2 n_2^2 + \theta_3 n_3^2}, \qquad \theta_j = \frac{1}{a_j^2}. \tag{4}$$

We introduce van der Pol transformation (cf. [8]) by setting $\mathbf{U}(t) = \mathbf{E}(-\Omega t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ is the "slow envelope" variable. We note that from (4) $\xi_n = 0$ for $n_3 = 0$, and therefore, $\mathbf{E}(-\Omega t) = \exp(-\Omega \mathbf{PJP}t)$ reduces to the identity operator on any barotropic (vertically averaged) field implying $\bar{\mathbf{U}} = \bar{\mathbf{u}}$. Equation (1) written in $\mathbf{u}$ variables has the form

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} = \mathbf{B}\left(\Omega t, \mathbf{u}, \mathbf{u}\right) + \mathbf{E}\left(\Omega t\right)\mathbf{F}, \qquad \mathbf{B}\left(\Omega t, \mathbf{u}, \mathbf{u}\right) = \mathbf{E}\left(\Omega t\right)\mathbf{B}\left(\mathbf{E}\left(-\Omega t\right)\mathbf{u}, \mathbf{E}\left(-\Omega t\right)\mathbf{u}\right). \quad (5)$$

The following limit equation [4–7] are associated with equation (5):

$$\partial_t \mathbf{w} - \nu\Delta\mathbf{w} = \tilde{\mathbf{B}}\left(\mathbf{w}, \mathbf{w}\right) + \tilde{\mathbf{F}}, \qquad \tilde{\mathbf{B}}\left(\mathbf{v}, \mathbf{v}\right) = \lim_{\Omega\to+\infty} \frac{1}{2\pi}\int_0^{2\pi} \mathbf{B}\left(\Omega s, \mathbf{v}, \mathbf{v}\right) ds. \quad (6)$$

In the above definition of $\tilde{\mathbf{B}}$ the arguments $\mathbf{v}$ are s-independent functions. From equations (5)–(6) it follows that the resonant interactions in $\tilde{\mathbf{B}}(\mathbf{w}, \mathbf{w})_n$ written in Fourier representation are present when $\pm\xi_k \pm \xi_m \pm \xi_n = 0$, $n = k + m$, where $\xi_n = \check{n}_3/|\check{n}|$ and similarly, for ξ_k and ξ_m. Thus, interactions in (6) are restricted to the resonant set of wave-numbers

$$K = \left\{\pm\frac{\check{k}_3}{|\check{k}|} \pm \frac{\check{m}_3}{|\check{m}|} \pm \frac{\check{n}_3}{|\check{n}|} = 0,\ n = k + m\right\}. \quad (7)$$

Equations (6) are called "2 1/2-dimensional" Navier-Stokes equations; both energy and helicity are conserved when $\nu = 0$, $\mathbf{F} = 0$. These equations are genuinely three-dimensional since they include all 3D modes but with restricted wave-number interactions in $\tilde{\mathbf{B}}(\mathbf{w}, \mathbf{w})$. They contain the classical two-dimensional Navier-Stokes equations as an invariant subsystem. This 2D subsystem corresponds to the interactions with $k_3 = m_3 = n_3 = 0$ in (7).

In [7], it was shown that $\overline{\tilde{\mathbf{B}}(\mathbf{w}, \mathbf{w})} = \tilde{\mathbf{B}}(\overline{\mathbf{w}}, \overline{\mathbf{w}})$, and therefore, solutions $\mathbf{w}(t, x_1, x_2, x_3) = (w_1, w_2, w_3)$ of the "2 1/2-dimensional" limit equations (6) split as $\mathbf{w} = \bar{\mathbf{w}} + \mathbf{w}_{\mathrm{dp}}$, where $\bar{\mathbf{w}}(t, x_1, x_2) = (\bar{w}_1, \bar{w}_2, \bar{w}_3)$ satisfies the 2D-3C Navier-Stokes equations (three components and dependence on two variables x_1, x_2):

$$\partial_t \bar{\mathbf{w}} + (\bar{\mathbf{w}} \cdot \nabla)\bar{\mathbf{w}} - \nu\Delta_h \bar{\mathbf{w}} = -\nabla_h \bar{q} + \bar{\mathbf{F}}, \qquad \nabla_h \cdot \bar{\mathbf{w}} = 0, \quad (8)$$

where ∇_h denotes the gradient in horizontal variables. The 'departure field' $\mathbf{w}_{\mathrm{dp}}(t, x_1, x_2, x_3)$ satisfies limit equations of the type

$$\partial_t \mathbf{w}_{\mathrm{dp}} - \nu\Delta\mathbf{w}_{\mathrm{dp}} = \mathbf{B}_2(\bar{\mathbf{w}}, \mathbf{w}_{\mathrm{dp}}) + \mathbf{B}_3(\mathbf{w}_{\mathrm{dp}}, \mathbf{w}_{\mathrm{dp}}) + \tilde{\mathbf{F}}_{\mathrm{dp}}. \quad (9)$$

If the force $\mathbf{F}(t, x)$ does not depend on Ω, then $\tilde{\mathbf{F}}_{\mathrm{dp}} = 0$. At the same time, $\tilde{\mathbf{F}}_{\mathrm{dp}} \neq 0$ may be generated by a resonant forcing as discussed in [7]; therefore, we include it in (9). Algebraic structure of $\mathbf{B}_2$ and $\mathbf{B}_3$ was already explicitly given in [4]. The catalytic operator $\mathbf{B}_2(\bar{\mathbf{w}}, \mathbf{w}_{\mathrm{dp}})$ was extensively studied in Section 5 of [7]. The only property of $\mathbf{B}_3$ from [4–7] used here is the estimate (15). In this paper, we present new estimates for the nonlinear "2 1/2 dimensional" operator $\mathbf{B}_3$ which do ensure global existence of strong solutions of the limit equation (9) and, consequently, equation (6) for *all* domain parameters. We note that pure three wave interactions correspond to $\xi_k\xi_m\xi_n \neq 0$ or, equivalently, $k_3 m_3 n_3 \neq 0$ and are associated with the term $\mathbf{B}_3$ in (9). We denote by $\tilde{K}$ the set of resonant wavevectors which do not correspond to pure three wave interactions, $\tilde{K} = K \cap \{k_3 m_3 n_3 = 0\}$. The set $\tilde{K}$ such that $n_3 \neq 0$ is the summation set for the catalytic operator $\mathbf{B}_2$ in (9); the set $K^* = K^*(\theta_1, \theta_2, \theta_3) = K \backslash \tilde{K}$ is the summation set for $\mathbf{B}_3$. The following theorem which will be proven below provides the main estimate for the resonant operator $\mathbf{B}_3$ for the 'worst' case (see comments below and Theorem 5).

THEOREM 2. *Let* $\mathbf{w} \in H^2$ *(Sobolev space of periodic vector fields with zero mean). Then the following estimate holds:*

$$|(\mathbf{B}_3(\mathbf{w},\mathbf{w}),\Delta\mathbf{w})| \leq \text{const}\,\|\mathbf{w}\|_2\,\|\mathbf{w}\|_1^2. \tag{10}$$

REMARK 1. Estimate (10) is of the same type as the classical estimate of Ladyzhenskaya [9] in the two-dimensional case with Dirichlet boundary conditions. For the periodic boundary conditions in 2D it is well known that the analog to the left-hand side of (10) is identically zero. Of course, in (10) the divergence free vector field $\mathbf{w}$ and the Sobolev spaces H^s are three-dimensional with space variables x_1, x_2, and x_3. From the estimate (10) we immediately obtain in a standard way (cf. [10,11]).

THEOREM 3. *Let* $\nu > 0, \|\mathbf{w}(0)\|_1 \leq M_1$, $\tilde{\mathbf{F}}_{\mathrm{dp}}$ *satisfies (11) with* $\alpha = 1$. *Then there exists a unique regular solution* $\mathbf{w}_{\mathrm{dp}}(t)$ *of "2 1/2-dimensional" Navier-Stokes equation (9),* $\|\mathbf{w}_{\mathrm{dp}}(t)\|_1 \leq M_1'(\nu, M_{F_1}, M_1, a_1, a_2, a_3)$ *for all* $t \geq 0$.

As corollary of Theorem 3 (with $\tilde{\mathbf{F}}_{\mathrm{dp}} = 0$*) on global existence of solutions of "21/2-dimensional" Navier-Stokes equations, convergence Theorems 6.3 and 8.2 in [7], local smoothing arguments and Theorems 2.3 and 2.4 in [12], we prove the theorem on global existence of strong solutions of 3D rotating Navier-Stokes equations for a nonresonant force.*

THEOREM 4. *Let aspect ratios* a_1, a_2, a_3 *be arbitrary and fixed. Let* $\nu > 0$, $\alpha > 1/2$, $\|\mathbf{U}(0)\|_\alpha \leq M_\alpha$ *and*

$$\sup_T \int_T^{T+1} \|\mathbf{F}\|_{\alpha-1}^2\,dt \leq M_{F_\alpha}^2, \qquad \forall T. \tag{11}$$

Let $\Omega \geq \Omega^*(a_1, a_2, a_3, M_\alpha, M_{F_\alpha}, \nu)$. *Then solutions of 3D rotating Navier-Stokes equations (1) are regular for all* $t \geq 0$ *and*

$$\|\mathbf{U}(t)\|_\alpha \leq M_\alpha', \qquad \forall t \geq 0. \tag{12}$$

From Theorem 4 we deduce Theorem 1 on regularization of weak Leray solutions and existence of global attractors for 3D rotating Navier-Stokes equations.

REMARK 2. The proof of Theorems 1 and 4 rely on the global regularity of the "2 1/2-dimensional" limit nonlinear Navier-Stokes equation (9) and techniques for convergence theorems as $\Omega \to \infty$ developed in [7,12]. The technique of bootstrapping regularity of solutions of three-dimensional Navier-Stokes equations by perturbation from limit equations has been done in various contexts: thin domains [13], helical flows [14]. In these previous works, limit equations are 2D Navier-Stokes equations for which global regularity is well known. In the present work, the limit equations are genuinely three-dimensional, but with restricted wave-number interactions in the nonlinear term. The existence and regularity theorems for those limit equations is nontrivial. Note that the smoothness conditions here are like in standard local regularity theorems and do not include technical conditions of BMN [4,7]. All restrictions on the domain parameters are also removed. Gallagher [15] in a work restricted to nonresonant domains (no consideration of "2 1/2-dimensional" nonlinear limit equations) imposes stronger restrictions on the force $\mathbf{F}$.

Now we outline the proof of Theorem 2.

LEMMA 1. *Let* $\chi(k,m,n)$ *be the characteristic function of some set* K^* *in* $(\mathbf{Z}^3)^3$ *such that* $\chi(k,m,n) = \chi(m,k,n) = \chi(k,n,m)$ *is symmetric. Let* $\alpha, \beta \geq 0$ *and*

$$\sup_n \sum_{\substack{k:k+m+n=0,\\ k\in\sum_i}} |\chi(k,m,n)|\,|k|^{-\alpha} \leq C_0 2^{i\beta} \tag{13}$$

for every $i = 0, 1, 2, \ldots,$ *where* $\sum_i = \{k = (k_1, k_2, k_3) \mid 2^i \le |k| < 2^{i+1}\}$, $|k| = \sqrt{k_1^2 + k_2^2 + k_3^2}$. *Then, for any* v_n *with* $v_{(0,0,0)} = 0$

$$\begin{aligned}&\sum_{k+m+n=0} |v_k||v_m||v_n|\chi(k,m,n)\\ &\le C\left(\sum_n |n|^\beta |v_n|^2\right)^{1/2}\left(\sum_k |k|^\alpha |v_k|^2\right)^{1/2}\left(\sum_m |v_m|^2\right)^{1/2},\end{aligned} \tag{14}$$

where $C = 6\sqrt{2C_0}$.

The proof of Lemma 1 relies on the technique of dyadic decomposition of Littlewood-Paley [16].

Proof of Theorem 2. *From [7] we have, using the skew-symmetry in the main part of the limit nonlinear operator* $\mathbf{B}_3$ *(three-wave interaction part in equation (9))*

$$|(\mathbf{B}_3(\mathbf{w}_{\mathrm{dp}}, \mathbf{w}_{\mathrm{dp}}), \Delta \mathbf{w}_{\mathrm{dp}})| \le c' \sum_{k+m+n=0} |k|\,|\mathbf{w}_{\mathrm{dp},k}|\,|m|\,|\mathbf{w}_{\mathrm{dp}.m}|\,|n|\,|\mathbf{w}_{\mathrm{dp},n}|\,\chi(k,m,n). \tag{15}$$

Here $\chi(k,m,n)$ is the characteristic function of the resonant set K^* of strict three-wave resonances (we put here for brevity $\theta_1 = \theta_2 = \theta_3 = 1$): $\pm k_3/|k| \pm m_3/|m| \pm n_3/|n| = 0$, $n + k + m = 0$, $k_3 m_3 n_3 \ne 0$. It is shown in Lemma 3.2 of [7] that this set lies in the set of solutions of equation $P(k,m,n) = 0$, where P is a polynomial of degree 12. From formulas of Lemma 3.2 of [7] follows that $P(k, -n-k, n)$ is a polynomial degree eight in k_3 and the coefficient at k_3^8 does not vanish for $n_3 \ne 0$. Let $k = (k', k_3)$, $k' = (k_1, k_2)$. For fixed k', n there are at most eight k_3 satisfying $P(k, -k-n, n) = 0$. Now we estimate the sum in (13) with $\alpha = 1$ as follows:

$$\sum_{2^i \le |k| < 2^{i+1}} \left(k_1^2 + k_2^2 + k_3^2\right)^{-1/2} \chi(k, -k-n, n) \le 8 + 8 \sum_{0 < |k'| < 2^{i+1}} \left(k_1^2 + k_2^2\right)^{-1/2} \le C_0 2^i,$$

where C_0 is an absolute constant. Therefore, the inequality (13) holds with $\alpha = \beta = 1$. Let $v_k = |k||\mathbf{w}_{\mathrm{dp},k}|$ and similarly, for m and n. Since $\|v\|_{1/2} = \|\mathbf{w}_{\mathrm{dp}}\|_{3/2}, \|v\|_0 = \|\mathbf{w}_{\mathrm{dp}}\|_1$, equations (14),(15) imply

$$\begin{aligned}|(\mathbf{B}_3(\mathbf{w}_{\mathrm{dp}}, \mathbf{w}_{\mathrm{dp}}), \Delta \mathbf{w}_{\mathrm{dp}})| &\le c' \sum_{k+m+n=0} |v_k||v_m||v_n|\chi(k,m,n)\\ &\le c'C\|v\|_{1/2}^2\|v\|_0 = c'C\|\mathbf{w}_{\mathrm{dp}}\|_1\|\mathbf{w}_{\mathrm{dp}}\|_{3/2}^2.\end{aligned} \tag{16}$$

After that we apply the interpolation inequality $\|\mathbf{w}_{\mathrm{dp}}\|_{3/2}^2 \le \|\mathbf{w}_{\mathrm{dp}}\|_1 \|\mathbf{w}_{\mathrm{dp}}\|_2$ and obtain from (16) the estimate (10) with $\mathbf{w} = \mathbf{w}_{\mathrm{dp}}$ (where the constant depends on a_1, a_2, a_3 in general case). This concludes the proof of Theorem 2.

In conclusion, we make some comments on the crucial role of the parameters $\theta_1 = 1/a_1^2$, $\theta_2 = 1/a_2^2$, $\theta_3 = 1/a_3^2$ for the properties of the dynamics, and in particular, for smoothness and error estimates. The operator $\mathbf{B}_3$ depends on $\{\theta_j\}_{j=1}^3$ discontinuously and solutions of the limit system with general initial data discontinuously depend on $\{\theta_j\}_{j=1}^3$ as proven in [7]. As solutions of the original Navier-Stokes equations depend on the domain parameters continuously, the convergence to solutions of the limit equations cannot be uniform. The estimates presented above are for the worst case and better estimates can be obtained for generic values of the domain parameters. Dependence of solutions to the limit equations on the domain parameters $\{\theta_j\}_{j=1}^{j=3}$ can be understood studying the geometry of resonant curves. For $k_3 m_3 n_3 \ne 0$, the resonant relation in equation (7) determining the set $K^* = K^*(\theta_1, \theta_2, \theta_3) = K \backslash \tilde{K}$ of three wave resonances can be written in the form

$$\pm \frac{1}{\sqrt{\theta_3 + \theta_2 b_k + \theta_1 d_k}} \pm \frac{1}{\sqrt{\theta_3 + \theta_2 b_m + \theta_1 d_m}} \pm \frac{1}{\sqrt{\theta_3 + \theta_2 b_n + \theta_1 d_n}} = 0, \tag{17}$$

with $b_k = k_2^2/k_3^2$, $d_k = k_1^2/k_3^2$, etc. Since this equation is homogeneous in $\theta_1, \theta_2, \theta_3$, we may put $\theta_1 = 1$. If the triplet k, m, n satisfies equation (17) so does the triplet $\lambda_1 k$, $\lambda_2 m$, $\lambda_3 n$. Therefore, for θ_j fixed, a solution of equation (17) is a triplet S of straight lines. The condition $k + m = n$ implies that all three lines lie in the same plane p_{kmn}. Now we consider (17) as equation for θ_2, θ_3 (recall that $\theta_1 = 1$) parametrized by a triplet S of coplanar straight lines through k, m, n in $\mathbf{Z}^3$. A detailed analysis shows that there are two essentially different cases. The first case (reducible) is when the plane p_{kmn} is orthogonal to one of three coordinate planes in $\mathbf{Z}^3$. In this case, equation (17) is satisfied by a straight line in θ_2, θ_3 plane. In the opposite case, equation (17) defines an irreducible algebraic curve Γ in θ_2, θ_3 plane. This curve determines uniquely four (related by reflections) triplets $S(\Gamma)$ of straight lines in $\mathbf{Z}^3$. For every point in θ_2, θ_3-plane, we denote by $N_{\mathrm{r}}(\theta_2, \theta_3) \leq \infty$ the number of different straight lines through it (reducible case). For irreducible curves through the point θ_2, θ_3, we consider for every line $\ell \in \mathbf{Z}^3$ such subset of the set of irreducible curves Γ, $(\theta_2, \theta_3) \in \Gamma$, that ℓ or its reflections belong to $S(\Gamma)$; the number of such curves we denote by $N_{\mathrm{ir}}(\theta_2, \theta_3, \ell)$ and the multiplicity $N_{\mathrm{ir}}(\theta_2, \theta_3) = \sup_\ell N_{\mathrm{ir}}(\theta_2, \theta_3, \ell)$. When $N_{\mathrm{ir}}(\theta_2, \theta_3) \neq \infty$, $N_{\mathrm{r}}(\theta_2, \theta_3) \leq \infty$ nonlinearity in the limit equation (6) admits much better estimate than in the case $N_{\mathrm{ir}}(\theta_2, \theta_3) \leq \infty, N_{\mathrm{r}}(\theta_2, \theta_3) \leq \infty$ considered in Theorem 2.

THEOREM 5. *Let* $\mathbf{w} \in H^2$, $N_{\mathrm{ir}}(\theta_2, \theta_3) < \infty$, $N_{\mathrm{r}}(\theta_2, \theta_3) \leq \infty$. *Then*

$$|\mathbf{B}_3(\mathbf{w}, \mathbf{w}), \Delta \mathbf{w})| \leq \frac{\nu}{2} \|\mathbf{w}\|_2^2 + \frac{C N_{\mathrm{ir}}}{\nu} \|\mathbf{w}\|_1^2 \|\mathbf{w}\|_0^2 + \frac{C(N_{\mathrm{r}})}{\nu^3} \|\mathbf{w}\|_1^2 \|\mathbf{w}\|_0^4,$$

where ($C_r(N_r) = 1$ *for* $N_{\mathrm{r}} \geq 1$, *(including* $N_{\mathrm{r}} = \infty$*) and* $C_r(N_{\mathrm{r}}) = 0$ *for* $N - \mathrm{r} = 0$.

Note that this inequality implies an algebraic dependence on ν^{-1} in estimates of solutions of the Navier-Stokes system, whereas (10) implies exponential.

REMARK 3. Let $\nu = 0$, $\mathbf{F} = 0$ in (1), that is, we consider the 3D rotating Euler equation. One can show that if $N_{\mathrm{ir}}(\theta_2, \theta_3) \leq 1$, $N_{\mathrm{r}}(\theta_2, \theta_3) = 0$, then the "2 1/2-dimensional" Euler system for $\mathbf{w}_{\mathrm{dp}}$ splits into finite-dimensional systems of nonlinear ODEs and the theorem on long-time regularity of solutions of 3D rotating Euler equation similar to [5] holds. This result generalizes the corresponding theorems in [5,7] where it was assumed that $N_{\mathrm{ir}}(\theta_2, \theta_3) = N_{\mathrm{r}}(\theta_2, \theta_3) = 0$.

REFERENCES

1. S.L. Sobolev, Ob odnoi novoi zadache matematicheskoi fiziki, *Izvestiia Akademii Nauk SSSR, Ser. Matematicheskaia* **18** (1), 3–50, (1954).
2. H. Poincaré, Sur la précession des corps déformables, *Bull. Astronomique* **27**, 321–356, (1910).
3. V.I. Arnold and B.A. Khesin, Topological methods in hydrodynamics, *Applied Mathematical Sciences* **12**, Springer, (1997).
4. A. Babin, A. Mahalov and B. Nicolaenko, Long-time averaged Euler and Navier-Stokes equations for rotating fluids, In *Structure and Dynamics of Nonlinear Waves in Fluids*, (Edited by K. Kirchgässner and A. Mielke), pp. 145–157, World Scientific, (1995).
5. A. Babin, A. Mahalov and B. Nicolaenko, Global splitting, integrability and regularity of 3D Euler and Navier-Stokes equations for uniformly rotating fluids, *European J. Mechanics B/ Fluids* **15**, 291–300, (1996).
6. A. Babin, A. Mahalov and B. Nicolaenko, Resonances and regularity for Boussinesq equations, *Russian J. Math. Physics* **4** (4), 417–428, (1996).
7. A. Babin, A. Mahalov and B. Nicolaenko, Global regularity and integrability of 3D Euler and Navier-Stokes equations for uniformly rotating fluids, *Asymptotic Analysis* **15**, 103–150, (1997).
8. V.I. Arnold, *Ordinary Differential Equations*, MIT Press, Cambridge, MA, (1973).
9. O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Second edition, Gordon and Breach, New York, (1969).
10. A.V. Babin and M.I. Vishik, *Attractors of Evolution Equations*, North-Holland, (1992).

11. P. Constantin and C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, (1988).
12. J. Avrin, A. Babin, A. Mahalov and B. Nicolaenko, On regularity of solutions of 3D Navier-Stokes equations, *Applicable Analysis* **71**, 197–214, (1999).
13. G. Raugel and G. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *J. Amer. Math. Soc.* **6** (3), 503–568, (1993).
14. A. Mahalov, S. Leibovich and E.S. Titi, Invariant helical subspaces for the Navier-Stokes equations, *Arch. for Rational Mech. and Anal.* **112** (3), 193–222, (1990).
15. I. Gallagher, Applications of Schochet's methods to parabolic equations, *J. Math. Pures Appl.* **77**, 989–1054, (1998).
16. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, (1970).
17. V.I. Arnold, Small denominators. I. Mappings of the circumference onto itself, *Amer. Math. Soc. Transl. Ser. 2* **46**, 213–284, (1965).